

Thermodynamics of a charged scalar field in disordered media

E. Arias,^{1,*} G. Krein,^{2,†} G. Menezes,^{2,‡} and N. F. Svaiter^{1,§}

¹*Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, RJ, Brazil*

²*Instituto de Física Teórica, Universidade Estadual Paulista*

Rua Dr. Bento Teobaldo Ferraz 271 - Bloco II, 01140-070 São Paulo, SP, Brazil

We investigate the thermodynamics of a free relativistic charged scalar field in the presence of weak disorder. We consider nonstatic quenched disorder which couples linearly to the mass of the scalar field. After performing noise averages over the free energy of the system, we find that disorder increases the critical temperature for Bose-Einstein condensation at finite density.

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I. INTRODUCTION AND MOTIVATION

Disorder plays an important role in the critical behavior of second order phase transitions [1]. The relevance of disorder in the criticality can be assessed qualitatively using the critical exponent of the specific heat α for the disorder-free system [2]; when $\alpha > 0$ (the specific heat diverges at the critical point), the critical behavior of the disordered system is changed, when $\alpha < 0$ (the specific heat is finite), disorder has no effect on the critical behavior. On the other hand, at low temperatures quantum fluctuations may compete with the random fluctuations. For example, the spin-glass ordered ground state – a disorder strongly correlated system – might be destroyed by quantum fluctuations. Recently, Carleo *et al.* [3] demonstrated the existence of Bose-Einstein condensation in quantum glasses due to frustration. In the present paper we focus on the interplay between quantum and random fluctuations in a relativistic charged scalar field theory with a finite chemical potential. Disorder in relativistic Bose-Einstein condensation seems not to have been considered in the literature, contrary to the case of non-relativistic case, where it has been under intensive study since early seminal works [4–6].

Disorder has a decisive influence on the zero-temperature phase diagram of non-relativistic Bose systems. As emphasized by the literature, there is a quantum phase transition for such systems from an Mott insulating to a conducting phase. Since no pure Bose system can be a normal conducting fluid at zero temperature, the conducting-insulator transition must correspond to the onset of superfluidity. As shown in Ref. [6], this scenario is changed dramatically in the presence of a random potential. For the case of a Gaussian colored noise, a Bose glass phase also arises and the transition to superfluidity only occurs from this third phase, never directly from the Mott insulator. The introduction of a random potential in such systems may also imply the destruction

of the superfluidity phase, as discussed in Refs. [5, 7]. In particular, a recent study by Lopatin and Vinokur [8], employing the replica method, found a negative shift in the condensation temperature of a dilute Bose gas due to static disorder – see also Refs. [9, 10].

There is an extensive literature on relativistic Bose-Einstein condensation (RBEC) following the pioneering works of Refs. [11–16], which discussed RBEC in flat space-times, and Refs. [17–20], which discussed RBEC in curved space-times. While relativistic Bose-Einstein condensates are not yet realizable in controllable experiments like their non-relativistic counterparts, they do relate to observable and experimentally accessible phenomena. One example, of immense current interest, concerns the condensation dynamics in relativistic quantum field theories where creation and annihilation of particles play crucial role, like in far-from-equilibrium stages of the early Universe and in experiments with relativistic heavy-ion collisions [21]. There is also the possibility of Bose-Einstein condensation of pions and kaons [22–25] in neutron stars. The condensation of these mesons will affect the equation of state of matter in the interior of the star, which has direct consequences on the observable mass-radius relation of the star, and will also impact the early evolution of the neutron star.

In real physical situations, the presence of some sort of disorder in the system is unavoidable. The disorder can be due to uncontrollable disturbances external to the system, or due to an incomplete treatment of degrees of freedom associated with fields that couple to the field of interest. As with nonrelativistic Bose-Einstein condensates of condensed matter physics, one expects that disorder will impact the critical behavior of relativistic Bose-Einstein condensation. The present study is a first step toward a systematic study of disorder in relativistic quantum field theory models, in that we focus on a non-interacting charged scalar field at finite temperature in the presence of nonstatic randomness (the condition for staticity will be properly defined shortly). Our model is a kind of generalization of the scalar Landau-Ginzburg theory, where the quenched disorder is described by random fluctuations of the effective transition temperature [1].

The organization of this paper is as follows. In Section II we present our model. The disorder field couples

* enrike@cbpf.br

† gkrein@ift.unesp.br

‡ gsm@ift.unesp.br

§ nfuxsvai@cbpf.br

to charged scalar field via the mass term of scalar field, just as in the random-temperature Landau-Ginzburg model. We consider weak disorder and implement a perturbative expansion for the free energy as power series expansion in the strength of the disorder field. In Section III we study the thermodynamics properties of the free relativistic Bose gas at finite density with randomness. In Section IV we obtain the critical temperature in the presence of random fluctuations. Conclusions and Perspectives are presented in Section V. The paper includes an Appendix containing details of renormalization of the propagators. Throughout the paper we employ units with $\hbar = c = k_B = 1$.

II. SCALAR FIELD AT FINITE TEMPERATURE IN DISORDERED MEDIA

We are interested in studying the effects of randomness on a charged scalar field φ of mass m in equilibrium with a thermal reservoir at temperature T . We employ the imaginary time formalism of Matsubara [26] to write the partition function of the model in the grand canonical ensemble as [13, 27]

$$Z = [N(\beta)]^2 \int [D\varphi][D\varphi^*] e^{S[\varphi, \varphi^*]}, \quad (1)$$

where the action $S[\varphi, \varphi^*]$ reads

$$S[\varphi, \varphi^*] = \int_0^\beta d\tau \int_V d\mathbf{x} \left[(\partial_t + i\mu) \varphi^* (\partial_t - i\mu) \varphi - \nabla \varphi^* \nabla \varphi - m^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2 \right], \quad (2)$$

where V is the volume of the system, $\beta = 1/T$, μ the chemical potential associated with the conserved charge, and $\partial_t = i \partial_\tau$. The field φ satisfies the Kubo-Martin-Schwinger [28] boundary condition $\varphi(\tau, \mathbf{x}) = \varphi(\tau + \beta, \mathbf{x})$. $N(\beta)$ is a β -dependent but μ -independent constant that comes from the integration over the canonical momentum conjugated to the field φ [27]. In this first study, we consider the non-interacting case $\lambda = 0$.

Next we consider the effects of a random noise source over the quantum matter field as in the random-temperature Landau-Ginzburg model, but generalized to the case of τ -dependent noise. That is, we perform the replacement $m^2 \rightarrow m^2(1 + \nu)$, where $\nu = \nu(\tau, \mathbf{x})$. The partition function given by Eq. (1) becomes replaced by

$$Z[\nu] = [N(\beta)]^2 \int [D\varphi][D\varphi^*] e^{S_T[\nu, \varphi, \varphi^*]}, \quad (3)$$

where

$$S_T[\nu, \varphi, \varphi^*] = S[\varphi, \varphi^*] + S_I[\nu, \varphi, \varphi^*], \quad (4)$$

with $S[\varphi, \varphi^*]$ given by Eq. (2) with $\lambda = 0$ and the contribution due to disorder is given by a random-potential

term

$$S_I[\nu, \varphi, \varphi^*] = -m^2 \int_0^\beta d\tau \int_V d\mathbf{x} \nu(\tau, \mathbf{x}) \varphi^*(\tau, \mathbf{x}) \varphi(\tau, \mathbf{x}). \quad (5)$$

The physical picture is that the random fluctuations describe average effects of external disturbances on the system or of degrees of freedom associated with fields that have been integrated out formally. Although similar to time-dependent noise [29], a τ dependence as in above should be understood as arising naturally when integrating out fields in favor of effective interactions of φ . Thence, we are not considering a perfect correlation of the disorder in the τ direction. This is in contrast to the anisotropic case where randomness depends only on spatial variables. In addition, hereafter by “static noise” we mean that one is considering a set of noises such that random fluctuations do not depend on τ . In this way, in the case studied in this paper one may qualify the τ -dependent noise as being nonstatic.

The standard procedure to study Bose-Einstein condensation is to separate from φ the constant zero mode $\langle \varphi \rangle \equiv \xi$:

$$\varphi = \xi + \chi, \quad (6)$$

where χ is a complex field with no zero mode. The χ field is written in terms of real and imaginary parts as

$$\chi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2), \quad (7)$$

so that the action can be written as

$$S_T[\nu, \chi_1, \chi_2, \xi] = -\beta V U(\xi) + S_0[\chi_1, \chi_2] + S_I[\nu, \chi_1, \chi_2, \xi], \quad (8)$$

with the classical potential

$$U(\xi) = (m^2 - \mu^2)\xi^2, \quad (9)$$

and the quadratic part

$$S_0[\chi_1, \chi_2] = -\frac{1}{2} \int_0^\beta d\tau \int_V d\mathbf{x} \left(\partial_\tau \chi_1 \partial_\tau \chi_1 + \nabla \chi_1 \nabla \chi_1 + (m^2 - \mu^2)\chi_1^2 + (m^2 - \mu^2)\chi_2^2 + \partial_\tau \chi_2 \partial_\tau \chi_2 + \nabla \chi_2 \nabla \chi_2 - 2i\mu(\chi_2 \partial_\tau \chi_1 - \chi_1 \partial_\tau \chi_2) \right). \quad (10)$$

In Eq. (10) we neglected the linear terms in the field χ , because their contributions will be proportional to terms like $\chi(\mathbf{p} = 0) = 0$. In turn, the random contribution is given by

$$S_I[\nu, \chi_1, \chi_2, \xi] = -m^2 \int_0^\beta d\tau \int_V d\mathbf{x} \left\{ \xi^2 \nu(\tau, \mathbf{x}) - \frac{1}{2} \nu(\tau, \mathbf{x}) [\chi_1^2(\tau, \mathbf{x}) + \chi_2^2(\tau, \mathbf{x})] - \sqrt{2}\xi \nu(\tau, \mathbf{x}) \chi_1(\tau, \mathbf{x}) \right\}. \quad (11)$$

Here we consider the random function $\nu(\tau, \mathbf{x})$ as a Gaussian distribution given by

$$P[\nu] = p_0 \exp\left(-\frac{1}{2\sigma^2} \int d^d x (\nu(x))^2\right). \quad (12)$$

where $x = (\tau, \mathbf{x})$ and p_0 is the normalization constant of the distribution. The quantity σ^2 is a parameter associated with the intensity of the disorder. We will denote the mean value over the random variable as $\overline{(\cdots)}$, defined by

$$\overline{A[\nu]} = \int [D\nu] P[\nu] A[\nu], \quad (13)$$

with $A[\nu]$ being any functional of ν . From Eq. (12), we have a white noise with two-point correlation function given by

$$\overline{\nu(\tau, \mathbf{x})\nu(\tau', \mathbf{x}')} = \sigma^2 \delta(\tau - \tau') \delta^3(\mathbf{x} - \mathbf{x}'). \quad (14)$$

As well known, it follows from the Gaussian distribution that

$$\overline{\nu(x_1) \cdots \nu(x_{2n+1})} = 0, \quad (15)$$

$$\overline{\nu(x_1) \cdots \nu(x_{2n})} = \sum_{\text{pair comb. pairs}} \prod \overline{\nu(x_j)\nu(x_k)}, \quad (16)$$

where n is an integer.

We are interested in calculating the critical temperature of condensation. We follow closely the path used for the noiseless case [30], in that the critical temperature is determined for a fixed charge density while simultaneously fixing the variational parameter ξ . Noise average is taken into account using Eq. (13), with A being the charge density $\rho = Q/V$. Specifically, the density is the derivative of the free-energy with respect to μ , and therefore:

$$\begin{aligned} \bar{\rho} &= \int [D\nu] P[\nu] \rho = \int [D\nu] P[\nu] \frac{1}{\beta V} \frac{\partial \ln Z_R[\nu]}{\partial \mu} \\ &= \frac{1}{\beta V} \frac{\partial}{\partial \mu} \int [D\nu] P[\nu] \ln Z_R[\nu] \\ &= \frac{1}{\beta V} \frac{\partial \overline{\ln Z_R[\nu]}}{\partial \mu}. \end{aligned} \quad (17)$$

Note that we are considering a situation where one has to deal with two kinds of averages, namely thermal averages and noise averages, which are not treated on the same footing. This can be justified when the characteristic time scale of the change in disorder is much larger than the time of observation of phenomena of interest.

Equation (17) requires a method to evaluate the average over noise realizations of the free energy. For static noise and arbitrary noise intensities the replica-trick is widely used [1]. Here we consider the weak-noise limit and use a perturbative approach [31, 32], in that one expands the partition function in a power series in the noise ν . This will be discussed in the next Section.

III. THE PARTITION FUNCTION AND NOISE AVERAGE OF THE FREE ENERGY

In the weak-disorder limit, the partition function in Eq. (3) can be expanded in a power series in S_I :

$$Z[\nu] = (N(\beta))^2 \int [d\chi_1][d\chi_2] e^{-\beta V U(\xi) + S_0} \sum_{n=0}^{\infty} \frac{S_I^n}{n!}, \quad (18)$$

where $S_I = S_I[\nu, \chi_1, \chi_2, \xi]$ and $S_0 = S_0[\chi_1, \chi_2]$. Taking the logarithm of both sides we get

$$\ln Z[\nu] = -\beta V U(\xi) + \ln Z_0 + \ln Z_I[\nu], \quad (19)$$

where the disorder-free part is given by

$$\ln Z_0 = \ln \left[(N(\beta))^2 \int [D\chi_1][D\chi_2] e^{S_0} \right], \quad (20)$$

and the corrections due to disorder are given by

$$\ln Z_I[\nu] = \ln \left(1 + \sum_{n=1}^{\infty} \frac{\langle S_I^n \rangle}{n!} \right). \quad (21)$$

Here the averages $\langle (\cdots) \rangle$ are defined using the unperturbed ensemble represented by the action S_0 :

$$\langle (\cdots) \rangle = \frac{\int [D\chi_1][D\chi_2] (\cdots) e^{S_0}}{\int [D\chi_1][D\chi_2] e^{S_0}}. \quad (22)$$

We now proceed to evaluate Eq. (19) up to second order. We start with the zeroth-order term, given by Eq. (20). This quantity is calculated explicitly in Appendix A and the result is

$$\begin{aligned} \ln Z_0 &= -V \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\beta \omega + \ln \left(1 - e^{-\beta(\omega(\mathbf{p}) - \mu)} \right) \right. \\ &\quad \left. + \ln \left(1 - e^{-\beta(\omega(\mathbf{p}) + \mu)} \right) \right], \end{aligned} \quad (23)$$

where $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$. Next, expanding Eq. (21) up to second order in the noise field, one obtains

$$\ln Z_I[\nu] = \langle S_I \rangle + \frac{1}{2} \left(\langle S_I^2 \rangle - \langle S_I \rangle^2 \right), \quad (24)$$

where S_I is given by Eq. (11). Let us now consider the stochastic average of such a contribution. From Eq. (15), we have that $\overline{\langle S_I \rangle} = 0$. The other terms are obtained using Eqs. (11) and (14):

$$\begin{aligned} \overline{\ln Z_I[\nu]} &= m^4 \sigma^2 \int_0^\beta d\tau \int_V d\mathbf{x} \left[\xi^2 \langle \chi_1^2 \rangle + \frac{1}{8} \left(\langle \chi_1^4 \rangle \right. \right. \\ &\quad \left. \left. + \langle \chi_2^4 \rangle - \langle \chi_1^2 \rangle^2 - \langle \chi_2^2 \rangle^2 \right) \right. \\ &\quad \left. + 2 \langle \chi_1^2 \chi_2^2 \rangle - 2 \langle \chi_1^2 \rangle \langle \chi_2^2 \rangle \right]. \end{aligned} \quad (25)$$

The derivation of the ensemble averages in Eq. (25) is outlined in Appendix B, and the result is

$$\begin{aligned} \overline{\ln Z_I[\nu]} = & (\beta V) m^4 \sigma^2 \left[\xi^2 \frac{1}{\beta} \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3} \mathcal{D}_{11}^0(\omega_n, \mathbf{p}) \right. \\ & + \frac{1}{4} \left(\frac{1}{\beta} \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3} \mathcal{D}_{11}^0(\omega_n, \mathbf{p}) \right)^2 \\ & \left. + \frac{1}{4} \left(\frac{1}{\beta} \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3} \mathcal{D}_{22}^0(\omega_n, \mathbf{p}) \right)^2 \right], \quad (26) \end{aligned}$$

where $\mathcal{D}_{ij}^0(\omega_n, \mathbf{p})$, $i, j = 1, 2$ are the zero-order propagators of the fields χ_j . Since the propagators have divergent vacuum contributions, Eq. (26) must be carefully regularized. The renormalization of the propagators is discussed in Appendix C. After carrying out such a procedure we get

$$\overline{\ln Z_I[\nu]} = (\beta V) \frac{m^4 \sigma^2}{2} [\Pi_m(2\xi^2 + \Pi_m) - \Pi_v^2], \quad (27)$$

where the quantities Π_v and $\Pi_m(\beta, \mu)$ are obtained in Appendix C; they are given by

$$\Pi_v = \frac{1}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega(\mathbf{p})}, \quad (28)$$

and

$$\begin{aligned} \Pi_m = \Pi_m(\beta, \mu) = & \frac{1}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega(\mathbf{p})} \left[\frac{1}{e^{\beta(\omega(\mathbf{p})-\mu)} - 1} \right. \\ & \left. + \frac{1}{e^{\beta(\omega(\mathbf{p})+\mu)} - 1} \right]. \quad (29) \end{aligned}$$

The last term in Eq. (27) is a divergent contribution to the zero-point energy density. Since in non-gravitational physics only energy differences are measurable, one can normalize the vacuum energy density at zero and then the renormalized $\ln Z$ up to second order in the random noise will be given by:

$$\begin{aligned} \frac{1}{\beta V} \overline{\ln Z_R[\nu]} = & [\mu^2 - m^2 + m^4 \sigma^2 \Pi_m(\beta, \mu)] \xi^2 \\ & - \frac{1}{\beta} \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\ln(1 - e^{-\beta(\omega(\mathbf{p})-\mu)}) \right. \\ & \left. + \ln(1 - e^{-\beta(\omega(\mathbf{p})+\mu)}) \right] + \frac{m^4 \sigma^2}{2} \Pi_m^2(\beta, \mu). \quad (30) \end{aligned}$$

Notice that thermal contributions arising from the corrections due to random fluctuations have a similar qualitative behavior for large (or small) β as the unperturbed part. In addition, notice that the momenta integrals are convergent for $|\mu| \leq m$. In the next Section we discuss the determination of the critical temperature.

IV. THE CRITICAL TEMPERATURE

As discussed in Ref. [30], the parameter ξ is not determined *a priori* and it should be treated as a variational parameter, related to the charge carried by the

condensed particles. At fixed β and μ , the free energy is an extremum with respect to variations of ξ . The derivative of the above equation with respect to ξ implies that $\xi = 0$ unless

$$\mu^2 - m^2 + m^4 \sigma^2 \Pi_m(\beta, \mu) = 0.$$

Considering a power series expansion for μ in the effective coupling $m^4 \sigma^2$ we have, at first order

$$|\mu| = [m^2 - m^4 \sigma^2 \Pi_m(\beta, m)]^{1/2}, \quad (31)$$

or, since we are working in the weak-disorder limit:

$$|\mu| = m \left[1 - \frac{m^4 \sigma^2}{2m^2} \Pi_m(\beta, m) \right]. \quad (32)$$

However, like in the noiseless case, this variational condition does not determine the parameter ξ . In order to fix its value the appropriate route is to calculate the charge density $\rho = Q/V$, which will also enable the determination of the critical temperature. As discussed previously, the charge density is given by

$$\bar{\rho} = \frac{1}{\beta V} \left(\frac{\partial \overline{\ln Z_R[\nu]}}{\partial \mu} \right)_{\mu=m'}, \quad (33)$$

where

$$m' = m \left[1 - \frac{m^4 \sigma^2}{2m^2} \Pi_m(\beta, m) \right]. \quad (34)$$

Using Eq. (30), the above expression becomes:

$$\bar{\rho} = \left[2m' + \frac{m^4 \sigma^2 \beta}{4} \Lambda(\beta, m') \right] \xi^2 + \rho^*(\beta, m') + \rho_I(\beta, m'), \quad (35)$$

where the unperturbed thermal contribution is

$$\begin{aligned} \rho^*(\beta, m') = & \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\frac{1}{e^{\beta(\omega(\mathbf{p})-m')} - 1} \right. \\ & \left. - \frac{1}{e^{\beta(\omega(\mathbf{p})+m')} - 1} \right], \quad (36) \end{aligned}$$

and the quantity $\rho_I(\beta, m')$ is defined as

$$\rho_I(\beta, m') = \frac{m^4 \sigma^2 \beta}{4} \Lambda(\beta, m') \Pi_m(\beta, m'), \quad (37)$$

with

$$\begin{aligned} \Lambda(\beta, m') = & \frac{1}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega(\mathbf{p})} \left[\frac{1}{\sinh^2(\beta(\omega(\mathbf{p}) - m')/2)} \right. \\ & \left. - \frac{1}{\sinh^2(\beta(\omega(\mathbf{p}) + m')/2)} \right]. \quad (38) \end{aligned}$$

If the density $\bar{\rho}$ is held fixed and the temperature is lowered, μ will increase until the point $\mu = m'$ is reached. If the temperature is lowered even further, then $\rho^* + \rho_I$ will be less than $\bar{\rho}$ and so

$$\xi^2 = 4 \left\{ \frac{\bar{\rho} - [\rho^*(\beta, m') + \rho_I(\beta, m')]}{8m' + m^4 \sigma^2 \beta \Lambda(\beta, m')} \right\}. \quad (39)$$

Hence, the critical temperature β_c^{-1} is determined implicitly by the equation given by

$$\bar{\rho} = \rho^*(\beta_c, m') + \rho_I(\beta_c, m'). \quad (40)$$

In order to clarify the influence of random fluctuations over thermal fluctuations, let us study the behavior of the critical temperature under two different circumstances. First, let us analyze the nonrelativistic limit, ($\beta m \gg 1$) of Eq. (40). In this limit the contributions from the antiparticles are negligible and, since $p \ll m$, we get $\omega(\mathbf{p}) \simeq m + \mathbf{p}^2/2m$ and $\omega^{-1}(\mathbf{p}) \simeq m^{-1}$. Therefore we get

$$\Pi_m(\beta, m') = \frac{1}{2m} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{e^{\beta[E(\mathbf{p}) + \gamma]} - 1}, \quad (41)$$

$$\rho^*(\beta, m') = 2m\Pi_m(\beta, m, \gamma), \quad (42)$$

and

$$\begin{aligned} \Lambda(\beta, m') &= \frac{1}{2m} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\sinh^2 \{\beta [E(\mathbf{p}) + \gamma] / 2\}} \\ &= -\frac{4}{\beta} \frac{\partial}{\partial \gamma} \Pi_m(\beta, m, \gamma), \end{aligned} \quad (43)$$

where $E(\mathbf{p}) = \mathbf{p}^2/2m$ and $m' = m - \gamma$, with

$$\beta\gamma = \beta \frac{m^4 \sigma^2}{2m} \Pi_m(\beta, m) \approx \frac{m^4 \sigma^2}{4(\beta m)^{1/2}} \frac{\zeta(3/2)}{(2\pi)^{3/2}}. \quad (44)$$

We performing the above integrals employing spherical coordinates, and using result [33]

$$\int_0^\infty dx \frac{x^{p-1}}{e^{rx} - q} = \frac{\Gamma(p)}{qr^p} \text{Li}_p(q), \quad (45)$$

where $\Gamma(x)$ is the usual Gamma function and $\text{Li}_n(z)$ is the polylogarithm function, whose series definition is given by

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}. \quad (46)$$

Therefore, inserting the results in Eq. (40) one has for the total charge density

$$\begin{aligned} \bar{\rho} &= \left(\frac{m}{2\pi\beta_c} \right)^{3/2} \text{Li}_{3/2}(e^{-\beta_c \gamma}) \\ &\times \left[1 + \frac{m^4 \sigma^2}{8\pi^2} \left(\frac{\pi}{2\beta_c m} \right)^{1/2} \text{Li}_{1/2}(e^{-\beta_c \gamma}) \right]. \end{aligned} \quad (47)$$

Since we are working in the weak-disorder limit, we may use the following power series expansion about $z = 0$, valid for all $s \neq 1, 2, 3, \dots$ [33]:

$$\text{Li}_s(e^{-z}) = \Gamma(1-s) z^{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \zeta(s-k)}{k!} z^k, \quad (48)$$

where $\zeta(x)$ is the usual Riemann zeta function. Therefore, keeping only terms up to order $m^4 \sigma^2$:

$$\bar{\rho} = \left(\frac{m}{2\pi\beta_c} \right)^{3/2} \zeta(3/2) \left[1 - \frac{1}{2} \eta^{1/2} - \frac{1}{2} \eta \right], \quad (49)$$

where

$$\eta = \eta(\beta_c, m) = \frac{m^4 \sigma^2}{(\beta_c m)^{1/2}} \frac{\pi}{(2\pi)^{3/2} \zeta(3/2)}. \quad (50)$$

Hence, the critical temperature reads

$$\beta_c^{-1} = \frac{2\pi}{m} \left(\frac{\bar{\rho}}{\zeta(3/2)} \right)^{2/3} \left[1 + \frac{1}{3} \eta^{1/2} + \frac{17}{36} \eta \right]. \quad (51)$$

Equation (51) is the expression for the critical temperature in the nonrelativistic limit. Solving such an equation by iteration one arrives at a power series expansion of β_c^{-1} in the effective coupling $m^4 \sigma^2$. Therefore, one has

$$\beta_c^{-1} = \beta_n^{-1} \left\{ 1 + \frac{1}{3} [\eta(\beta_n, m)]^{1/2} + \frac{17}{36} \eta(\beta_n, m) \right\}, \quad (52)$$

where the nonrelativistic critical temperature β_n^{-1} in the absence of disorder reads

$$\beta_n^{-1} = \frac{2\pi}{m} \left(\frac{\bar{\rho}}{\zeta(3/2)} \right)^{2/3}. \quad (53)$$

We see that the effect of disorder in the non-relativistic limit implies in a higher condensation temperature than the one in absence of disorder.

Now let us analyze the ultrarelativistic limit ($\beta m \ll 1$) of Eq. (40). In this limit we have $p \gg m$, so that one has $\omega(\mathbf{p}) \simeq p$. Then

$$\rho^*(\beta, m') \approx h(m') - h(-m'), \quad (54)$$

and

$$\begin{aligned} \rho_I(\beta, m') &\approx \frac{m^4 \sigma^2 \beta}{16} [F(m') + F(-m')] \\ &\times [G(m') - G(-m')], \end{aligned} \quad (55)$$

where

$$h(m') = \frac{1}{2\pi^2} \int_0^\infty dp \frac{p^2}{e^{\beta(p-m')} - 1}, \quad (56)$$

$$F(m') = \frac{1}{2\pi^2} \int_0^\infty dp \frac{p}{e^{\beta(p-m')} - 1}, \quad (57)$$

$$\begin{aligned} G(m') &= \frac{2}{\pi^2} \int_0^\infty dp \frac{p e^{\beta(p-m')}}{[e^{\beta(p-m')} - 1]^2} \\ &= \frac{4}{\beta} \frac{d}{dm'} F(m'), \end{aligned} \quad (58)$$

and

$$m' = m \left[1 - \frac{m^4 \sigma^2}{2m^2} \Pi_m(\beta, m) \right] \approx m \left\{ 1 - \frac{m^4 \sigma^2}{4m^2} [F(m) + F(-m)] \right\}. \quad (59)$$

First let us consider $F(m')$. We use Eq. (45) and the result [34]:

$$\text{Li}_n(e^{2\pi iz}) + (-1)^n \text{Li}_n(e^{-2\pi iz}) = -\frac{(2\pi i)^n}{n!} B_n(z), \quad (60)$$

where $B_n(z)$ are the Bernoulli polynomials, which have the following series representations [35]

$$B_{2n-1}(x) = \frac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n-1}}, \quad (61)$$

and

$$B_{2n}(x) = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2n}}. \quad (62)$$

Hence

$$F(m') + F(-m') = \frac{1}{\beta^2} B_2(x), \quad (63)$$

where $2\pi x = \beta m'$. In the same way we obtain for $G(m')$:

$$G(m') - G(-m') = -\frac{4i}{\pi\beta^2} B_1(x). \quad (64)$$

Finally, following similar steps as above we obtain for $h(m')$:

$$h(m') - h(-m') = \frac{4\pi i}{3\beta^3} B_3(x). \quad (65)$$

Therefore, inserting the results in Eq. (40), one obtains for the total charge density:

$$\bar{\rho} = \frac{4\pi i}{3\beta^3} B_3\left(\frac{\beta m'}{2\pi i}\right) - i \frac{m^4 \sigma^2}{4\pi\beta^3} B_1\left(\frac{\beta m'}{2\pi i}\right) B_2\left(\frac{\beta m'}{2\pi i}\right). \quad (66)$$

A more transparent expression can be obtained when we make use of the relations in Eqs. (61) and (62) for the Bernoulli polynomials. Since $\beta m \ll 1$ and we are working in the weak-disorder limit, one may Taylor expand the circular functions in such equations and neglect all terms of order $(\beta m')^2$ and higher. Inserting the results in Eq. (66) we get, up to order $m^4 \sigma^2$:

$$\bar{\rho} = \frac{m}{3\beta_c^2} \left[1 - \frac{m^4 \sigma^2}{16\pi^2} - \frac{m^4 \sigma^2}{24(\beta_c m)^2} \right]. \quad (67)$$

Hence the ultrarelativistic critical temperature in the weak-disorder limit is given by:

$$\beta_c^{-1} = \left(\frac{3\bar{\rho}}{m} \right)^{1/2} \left[1 + \frac{m^4 \sigma^2}{32\pi^2} + \frac{m^4 \sigma^2}{48(\beta_c m)^2} \right]. \quad (68)$$

This is the result for the critical temperature in the ultrarelativistic limit. As above, solving such an equation by iteration one arrives at a power series expansion of β_c^{-1} in the effective coupling $m^4 \sigma^2$. Therefore, at first order, one has

$$\beta_c^{-1} = \beta_u^{-1} \left[1 + \frac{m^4 \sigma^2}{32\pi^2} + \frac{m^4 \sigma^2}{48(\beta_u m)^2} \right], \quad (69)$$

where the ultrarelativistic critical temperature β_u^{-1} in the absence of disorder is given by

$$\beta_u^{-1} = \left(\frac{3\bar{\rho}}{m} \right)^{1/2}. \quad (70)$$

As in the nonrelativistic limit, we find a positive shift in the critical temperature due to random fluctuations.

V. CONCLUSIONS AND PERSPECTIVES

In this paper we investigated the effect of weak disorder on a free relativistic charged scalar field in thermal equilibrium with a reservoir. We studied the effect of coupling of a random field to the scalar field in the situation where Bose-Einstein condensation takes place. We considered a quenched disorder which couples linearly to the mass of the scalar field, just as in the random-temperature Landau-Ginzburg model. After performing noise averages of the free energy, we obtained the corrections to the critical temperature for the free Bose gas at finite density. We showed that the effect of the randomness is to increase the critical temperature in both nonrelativistic and ultrarelativistic limits. Naturally, one should keep in mind that this is a result valid for weak disorder and obtained in the framework of a perturbative expansion in the noise intensity. It remains to be seen if the same result is attainable with a nonperturbative calculation, e.g. using a replica trick. For static (τ -independent) noise, application of the replica-trick consists in the following [1]. Using the fact that $\ln Z = \lim_{n \rightarrow 0} (Z^n - 1)/n$, one has that $\overline{\ln Z[\nu]} = \lim_{n \rightarrow 0} (Z_n - 1)/n$, where $Z_n = \overline{Z^n[\nu]}$. The Z_n is interpreted as the partition function of a new system, formed from n statistically independent copies of the original system. The quenched free energy functional is defined as $F_q(h) \equiv F_q = \lim_{n \rightarrow 0} Z_n - 1/n$, showing that the quenched free energy functional can be calculated from a zero-component field theory.

To conclude, we remark on an important point with respect to the fact that random mass models generate effective interactions that mimic a negative coupling constant. Many authors claim that non-relativistic bosons only make sense in a random potential when they present repulsive interactions [6]. Nevertheless, there are many examples that justify the study of a free theory in the presence of random fluctuations. For instance, relativistic scalar field models with negative coupling constant were investigated in the literature and meaningful results

were obtained – see for example Refs. [36–41]. Based on the results obtained in Ref. [42], where it has been shown that the theory with a negative coupling constant develops a condensate, Arias *et al* [43] discussed the thermodynamics of a asymptotically free Euclidean self-interacting scalar field defined in a compact spatial region without boundaries. In addition, based on the results of Refs. [44, 45], we have discussed the effects of randomness over quantum fields in different physical scenarios. Ref. [31] proposed in condensed matter physics setting an analog model for fluctuations of the light cone. Also, a free massive scalar field in inhomogeneous random media was studied in [32]. After performing the averages over the random functions, the two- and four-point causal Green's function of the model were presented up to one-loop approximation. More recently, Refs. [46] and [47] investigated the influence of fluctuations of the event horizon on the transition rate of a two-level system which interact with a quantum field. The natural extension of such works is to consider the renormalized vacuum energy density associated with a quantum field in the presence of a stochastic ensemble of fluctuating geometries. This is under investigation by the authors.

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Appendix A: Calculation of the partition function

In this Appendix we calculate the logarithm of the free partition function given by Eq. (20). We start by introducing Fourier series to the fields χ_1 and χ_2 :

$$\chi_i(\tau, \mathbf{x}) = \left(\frac{\beta}{V}\right)^{1/2} \sum_n \sum_{\mathbf{p}} e^{i(\mathbf{p} \cdot \mathbf{x} + \omega_n \tau)} \chi_{i;n}(\mathbf{p}), \quad (\text{A1})$$

with $i = 1, 2$ and $\beta\omega_n = 2\pi n$ due to the constraint of periodicity $\chi_i(0, \mathbf{x}) = \chi_i(\beta, \mathbf{x})$ for all \mathbf{x} . Inserting this last result in the free field action given by Eq. (10) we obtain, after performing an integration by parts:

$$S_0 = -\frac{1}{2} \sum_{n, \mathbf{p}} \begin{pmatrix} \chi_{1;-n}(-\mathbf{p}) & \chi_{2;-n}(-\mathbf{p}) \end{pmatrix} \Theta \begin{pmatrix} \chi_{1;n}(\mathbf{p}) \\ \chi_{2;n}(\mathbf{p}) \end{pmatrix}, \quad (\text{A2})$$

where we discarded a total derivative term and we defined the matrix $\Theta = \Theta_n(\mathbf{p})$ as

$$\Theta_n(\mathbf{p}) = \beta^2 \begin{pmatrix} \omega_n^2 + \omega^2(\mathbf{p}) - \mu^2 & -2\mu\omega_n \\ 2\mu\omega_n & \omega_n^2 + \omega^2(\mathbf{p}) - \mu^2 \end{pmatrix}, \quad (\text{A3})$$

with $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$. Thus, using Eq. (A2) the logarithm of the free partition function now becomes

$$\ln Z_0 = \ln(N(\beta))^2 + \ln J(\beta, \mu) \quad (\text{A4})$$

where

$$\begin{aligned} J(\beta, \mu) = & \prod_n \prod_{\mathbf{p}} \int d\chi_{2;n}(\mathbf{p}) \exp \left\{ -\frac{1}{2} \left[\Theta_n(\mathbf{p}) \right]_{22} \chi_{2;n}(\mathbf{p}) \chi_{2;-n}(-\mathbf{p}) \right\} \\ & \times \int d\chi_{1;n}(\mathbf{p}) \exp \left\{ -\frac{1}{2} \left[\Theta_n(\mathbf{p}) \right]_{11} \chi_{1;n}(\mathbf{p}) \chi_{1;-n}(-\mathbf{p}) \right\} \\ & \times \exp \left\{ -\frac{1}{2} \left[\Theta_n(\mathbf{p}) \right]_{12} \chi_{2;n}(\mathbf{p}) \chi_{1;-n}(-\mathbf{p}) - \frac{1}{2} \left[\Theta_n(\mathbf{p}) \right]_{21} \chi_{1;n}(\mathbf{p}) \chi_{2;-n}(-\mathbf{p}) \right\}. \end{aligned} \quad (\text{A5})$$

Noting that $\chi_{i;-n}(-\mathbf{p}) = \chi_i^*(\mathbf{p})$, $i = 1, 2$, as required by the reality of the fields $\chi_i(\tau, \mathbf{x})$, the above integrals are just generic Gaussian integrals. Therefore

$$J(\beta, \mu) = [\det \Theta_n(\mathbf{p})]^{-1/2}, \quad (\text{A6})$$

where we have discarded an overall constant multiplicative factor. Inserting this last expression in Eq. (A4), we get:

$$\ln Z_0 = \ln(N(\beta))^2 - \frac{1}{2} \ln \det \Theta_n(\mathbf{p}), \quad (\text{A7})$$

where

$$\begin{aligned} \ln \det \Theta_n(\mathbf{p}) = & \ln \left\{ \prod_n \prod_{\mathbf{p}} \beta^2 [(\omega_n + i\mu)^2 + \omega^2(\mathbf{p})] \right\} \\ & + \ln \left\{ \prod_n \prod_{\mathbf{p}} \beta^2 [(\omega_n - i\mu)^2 + \omega^2(\mathbf{p})] \right\}, \end{aligned} \quad (\text{A8})$$

and, according to Ref. [27]

$$\ln(N(\beta)) = -V \ln \beta \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3}. \quad (\text{A9})$$

Hence

$$\begin{aligned} \ln Z_0 &= -\frac{1}{2}V \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3} \ln \{ \beta^2 [(\omega_n + i\mu)^2 + \omega^2] \} \\ &\quad -\frac{1}{2}V \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3} \ln \{ \beta^2 [(\omega_n - i\mu)^2 + \omega^2] \} \\ &\quad -2V \ln \beta \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3}, \end{aligned} \quad (\text{A10})$$

where $\omega = \omega(\mathbf{p})$ and we consider V to be large compared to all other physical lengths so we can replace the sum over \mathbf{p} with an integral. The frequency sums can be performed using standard procedures [48] and the result is given by Eq. (23).

Appendix B: Calculation of the corrections to the free energy

In this Appendix we evaluate the perturbative corrections to $\ln Z$ given by Eq. (25). For this purpose, let us present some basic derivations. We define the finite-temperature propagators in position space as [30]

$$\mathcal{D}_{ij}(\tau, \mathbf{x}; \tau', \mathbf{x}') = \langle \chi_i(\tau, \mathbf{x}) \chi_j(\tau', \mathbf{x}') \rangle, i, j = 1, 2. (\text{B1})$$

Due to translation invariance, the above propagators should depend only on $\mathbf{x} - \mathbf{x}'$ and $\tau - \tau'$. With the help of Eq. (A1), their Fourier transforms are

$$\begin{aligned} \mathcal{D}_{ij}(\omega_n, \mathbf{p}) &= \int_0^\beta d\tau \int_V d\mathbf{x} e^{-i(\mathbf{p} \cdot \mathbf{x} + \omega_n \tau)} \mathcal{D}_{ij}(\tau, \mathbf{x}) \\ &= \beta^2 \langle \chi_{i;n}(\mathbf{p}) \chi_{j;-n}(-\mathbf{p}) \rangle, \end{aligned} \quad (\text{B2})$$

where $\mathcal{D}_{ij}(\tau, \mathbf{x}) = \mathcal{D}_{ij}(\tau, \mathbf{x}; 0, 0)$. Alternatively, we note that the propagators can be expressed as functional derivatives of the partition function:

$$\mathcal{D}_{ij}(\omega_n, \mathbf{p}) = 2 (\mathcal{D}_{ij}^0)^2 \frac{\delta \ln Z[\nu]}{\delta \mathcal{D}_{ij}^0}. \quad (\text{B3})$$

Hence, with the help of Eq. (A4) the zero-order propagator $\mathcal{D}_{11}^0(\omega_n, \mathbf{p})$ is given by

$$\mathcal{D}_{11}^0(\omega_n, \mathbf{p}) = -2\beta^2 \frac{\delta \ln Z_0}{\delta [\Theta_n(\mathbf{p})]_{11}}. \quad (\text{B4})$$

Similar expressions hold for the other zero-order propagators. Using Eq. (A7) together with Eq. (A3), we get

$$\mathcal{D}_{11}^0(\omega_n, \mathbf{p}) = \frac{1}{2} [\mathcal{D}_+(\omega_n, \mathbf{p}) + \mathcal{D}_-(\omega_n, \mathbf{p})] \quad (\text{B5})$$

where

$$\mathcal{D}_\pm(\omega_n, \mathbf{p}) = \frac{1}{(\omega_n \pm i\mu)^2 + \omega^2(\mathbf{p})}. \quad (\text{B6})$$

In the same way:

$$\mathcal{D}_{22}^0(\omega_n, \mathbf{p}) = \frac{1}{2} [\mathcal{D}_+(\omega_n, \mathbf{p}) + \mathcal{D}_-(\omega_n, \mathbf{p})], \quad (\text{B7})$$

and

$$\begin{aligned} \mathcal{D}_{12}^0(\omega_n, \mathbf{p}) &= -\mathcal{D}_{21}^0(\omega_n, \mathbf{p}) \\ &= \frac{2\mu\omega_n}{[\omega_n^2 + (\omega(\mathbf{p}) + \mu)^2][\omega_n^2 + (\omega(\mathbf{p}) - \mu)^2]}. \end{aligned} \quad (\text{B8})$$

We may apply the above technique in order to calculate the various ensemble averages emerging in Eq. (25). Using Eqs. (A1) and (B2) one can easily show that

$$\int_0^\beta d\tau \int_V d\mathbf{x} \langle \chi_1^2(\tau, \mathbf{x}) \rangle = \sum_{n, \mathbf{p}} \mathcal{D}_{11}^0(\omega_n, \mathbf{p}). \quad (\text{B9})$$

Following the same strategy, we have that

$$\int_0^\beta d\tau \int_V d\mathbf{x} \langle \chi_1^4(\tau, \mathbf{x}) \rangle = \frac{3}{\beta V} \left[\sum_{n, \mathbf{p}} \mathcal{D}_{11}^0(\omega_n, \mathbf{p}) \right]^2, \quad (\text{B10})$$

and

$$\int_0^\beta d\tau \int_V d\mathbf{x} \langle \chi_1^2(\tau, \mathbf{x}) \rangle^2 = \frac{1}{\beta V} \left[\sum_{n, \mathbf{p}} \mathcal{D}_{11}^0(\omega_n, \mathbf{p}) \right]^2. \quad (\text{B11})$$

Similar expressions are valid for the field χ_2 . Then, using the same procedures as above, one has

$$\int_0^\beta d\tau \int_V d\mathbf{x} \langle \chi_1^2(\tau, \mathbf{x}) \rangle \langle \chi_2^2(\tau, \mathbf{x}) \rangle = \int_0^\beta d\tau \int_V d\mathbf{x} \langle \chi_1^2(\tau, \mathbf{x}) \chi_2^2(\tau, \mathbf{x}) \rangle = \frac{1}{\beta V} \left[\sum_{n, \mathbf{p}} \mathcal{D}_{11}^0(\omega_n, \mathbf{p}) \right]^2, \quad (\text{B12})$$

since $\mathcal{D}_{11}^0(\omega_n, \mathbf{p}) = \mathcal{D}_{22}^0(\omega_n, \mathbf{p})$ and $\sum_n \mathcal{D}_{12}^0(\omega_n, \mathbf{p}) = \sum_n \mathcal{D}_{21}^0(\omega_n, \mathbf{p}) = 0$. Inserting the above results in Eq. (25) and considering the large-volume limit leads us to expression (26). This is the contribution to $\ln Z$ due

to randomness up to second order in the noise. Nevertheless, this is not the final answer; as well known, the vacuum contribution to the propagators is divergent. Therefore, in order to present a finite result for $\ln Z$ one

must first renormalize $\mathcal{D}_{11}^0(\omega_n, \mathbf{p})$ and $\mathcal{D}_{22}^0(\omega_n, \mathbf{p})$. This we discuss in the next Appendix.

Appendix C: Renormalization of propagators

Here we examine the renormalization of the finite-temperature propagators $\mathcal{D}_{11}^0(\omega_n, \mathbf{p})$ and $\mathcal{D}_{22}^0(\omega_n, \mathbf{p})$. Again following Ref. [30] we define the self-energy $\Pi_1 = \Pi_1(\omega_n, \mathbf{p})$ with respect to the averaged propagator $\overline{\mathcal{D}_{11}}$ as

$$\overline{\mathcal{D}_{11}(\omega_n, \mathbf{p})} = (1 + \mathcal{D}_{11}^0 \Pi_1)^{-1} \mathcal{D}_{11}^0. \quad (\text{C1})$$

A similar expression holds for $\Pi_2 = \Pi_2(\omega_n, \mathbf{p})$ which is the self-energy with respect to the averaged propagator $\overline{\mathcal{D}_{22}}$. Hence, recalling Eqs. (B3) and (B4), we get, up to second order in ν :

$$(1 + \mathcal{D}_{11}^0 \Pi_1)^{-1} = 1 + 2\mathcal{D}_{11}^0 \frac{\delta \ln Z_I[\nu]}{\delta \mathcal{D}_{11}^0}, \quad (\text{C2})$$

and

$$(1 + \mathcal{D}_{22}^0 \Pi_2)^{-1} = 1 + 2\mathcal{D}_{22}^0 \frac{\delta \ln Z_I[\nu]}{\delta \mathcal{D}_{22}^0}. \quad (\text{C3})$$

Therefore, inserting Eq. (26) in the above equations and expanding their left-hand sides to first order yields the following expressions for the self-energies

$$\Pi_1(\omega_n, \mathbf{p}) = -2m^4 \sigma^2 \xi^2 - \frac{m^4 \sigma^2}{\beta} \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3} \mathcal{D}_{11}^0(\omega_n, \mathbf{p}), \quad (\text{C4})$$

and

$$\Pi_2(\omega_n, \mathbf{p}) = -\frac{m^4 \sigma^2}{\beta} \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3} \mathcal{D}_{22}^0(\omega_n, \mathbf{p}). \quad (\text{C5})$$

On the other hand

$$\frac{1}{\beta} \sum_n \int \frac{d\mathbf{p}}{(2\pi)^3} \mathcal{D}_{ii}^0(\omega_n, \mathbf{p}) = \frac{1}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega(\mathbf{p})} \left[1 + \frac{1}{e^{\beta(\omega(\mathbf{p})-\mu)} - 1} + \frac{1}{e^{\beta(\omega(\mathbf{p})+\mu)} - 1} \right], \quad (\text{C6})$$

for $i = 1, 2$, where we have performed the sums over the Matsubara frequencies in the usual way [48]. In this way we have

$$\Pi_1(\omega_n, \mathbf{p}) = -m^4 \sigma^2 [2\xi^2 + \Pi_v + \Pi_m(\beta, \mu)], \quad (\text{C7})$$

and

$$\Pi_2(\omega_n, \mathbf{p}) = -m^4 \sigma^2 [\Pi_v + \Pi_m(\beta, \mu)], \quad (\text{C8})$$

where we have defined

$$\Pi_v = \frac{1}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega(\mathbf{p})}, \quad (\text{C9})$$

and

$$\Pi_m(\beta, \mu) = \frac{1}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega(\mathbf{p})} \left[\frac{1}{e^{\beta(\omega(\mathbf{p})-\mu)} - 1} + \frac{1}{e^{\beta(\omega(\mathbf{p})+\mu)} - 1} \right]. \quad (\text{C10})$$

Since Π_v is a divergent quantity, in order to avoid physically meaningless results the following counterterm must be added to the original action:

$$\begin{aligned} \delta S &= -\delta m^2 \int_0^\beta d\tau \int_V d\mathbf{x} \varphi \varphi^* \\ &= -\frac{\delta m^2}{2} \int_0^\beta d\tau \int_V d\mathbf{x} (2\xi^2 + \chi_1^2 + \chi_2^2), \end{aligned} \quad (\text{C11})$$

where we have dropped terms linear in χ_1 and χ_2 . Treating this as an additional interaction, we see from Eq. (21) that to lowest order this counterterm contributes to $\ln Z_I$ as

$$-(\beta V) \delta m^2 \xi^2 - \frac{\delta m^2}{2} \int_0^\beta d\tau \int_V d\mathbf{x} (\langle \chi_1^2 \rangle + \langle \chi_2^2 \rangle). \quad (\text{C12})$$

The counterterm should be chosen so that

$$\delta m^2 - m^4 \sigma^2 \Pi_v = 0. \quad (\text{C13})$$

In this way we get a finite result for the propagators. Whence, collecting the above results, the contribution to $\ln Z$ up to second order in the noise will be

$$\overline{\ln Z_I[\nu]} = (\beta V) \frac{m^4 \sigma^2}{2} [\Pi_m(2\xi^2 + \Pi_m) - \Pi_v^2]. \quad (\text{C14})$$

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